The spectrum of a ring

The functor Spec lets us associate a geometric object to each ring.

Def: Let R be a ring. Let $\underline{\operatorname{Spec} R}$ denote the set of prime ideals of R.

We define a topology, the Zariski topology, on $\operatorname{Spec} R$ as follows:

For any subset $I \subseteq R$, $V(I) := \{P \in \operatorname{Spec} R \mid I \subseteq P\}$.

Note: $V(I) = V(\text{ideal gen. by } I)$, so we can restrict our definition to ideals.

In fact, by Section 2, $V(I) = V(\sqrt{I})$.

$V(I)$ are the closed sets of the Zariski topology.

Note $V(0) = \operatorname{Spec} R$ and $V(R) = \emptyset$.

Claim:

a.) $\bigcap V(I_\lambda) = V\left(\sum_\lambda I_\lambda\right)$

b.) $V(I) \cup V(J) = V(I \cap J) = V(IJ)$

Pf: a.) $P \in \bigcap V(I_\lambda) \iff P \supseteq \bigcup I_\lambda \iff P \in V\left(\sum_\lambda I_\lambda\right)$.

b.) $IJ \subseteq I \cap J$, so $V(IJ) \supseteq V(I \cap J)$.

If $P \in V(I) \cup V(J)$ then $P \supseteq I$ or J so $P \supseteq I \cap J$, so $V(I) \cup V(J) \subseteq V(I \cap J)$.

Now suppose $P \supseteq IJ$ but $P \not\supseteq I$. i.e. there is $u \in I$ but $u \notin P$. $\forall$ $v \in J$, $uv \in P$, so $v \in P$. Thus $J \subseteq P$. $\Rightarrow$ $V(I) \cup V(J) \supseteq V(IJ)$. $\square$

Note that a single point set $\{P\}$ is closed iff P is maximal. The subset of max'l ideals is <u>maxspec(R)</u>.

Geometrically, these are the points we usually consider.

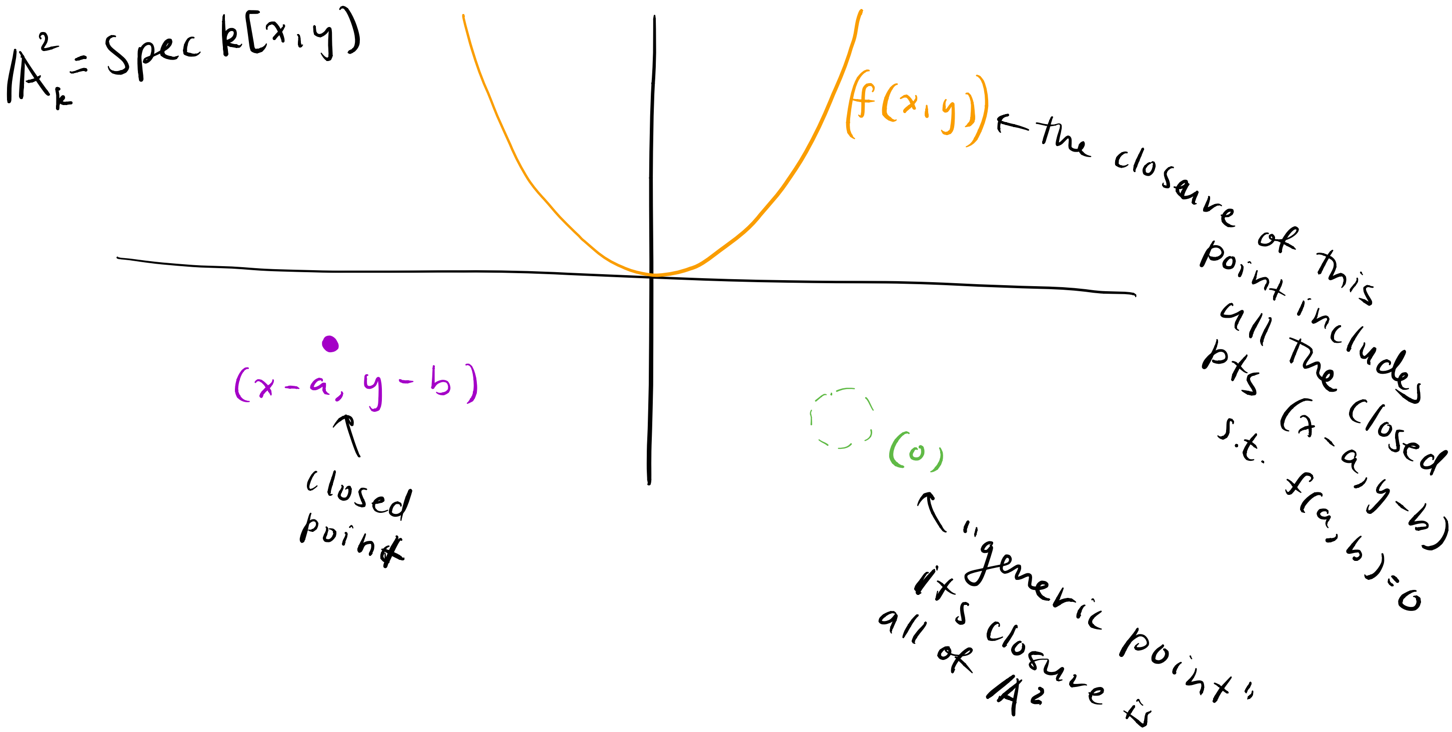
Ex: 1.) $\operatorname{Spec} k[x] = \{(f(x))\}$. f an irred. polynomial. If k is algebraically closed, these are in bijection w/ sets of points on k, and (0).

$\max\operatorname{Spec}(k[x]) =$ points of k. (More about this in the next section)

2.) $\max\operatorname{Spec} k[x,y] = \{(x-a, y-b) \mid a,b \in k\}$ $(k = \bar{k})$

so the closed points are in bijection w/ k^2.

The other points of spec correspond to curves defined by irreducible polynomials $f(x,y)$.



(More generally, $\mathbb{A}^n_k = \operatorname{Spec}(k[x_1, \dots, x_n])$)

Spec as a functor

Spec is a contravariant functor * from rings to topological spaces:

* HW: look up covariant + contravariant functors

If $\varphi : R \to S$ is a ring map, then define

$$\operatorname{Spec}(\varphi) : \operatorname{Spec}(S) \to \operatorname{Spec}(R)$$
$$P \longmapsto \varphi^{-1}(P)$$

This induced map is continuous, i.e. the preimage of a closed set is closed:

Claim: If $\varphi: R \to S$ is a ring map, and $I \subseteq R$, then $(\operatorname{Spec} \varphi)^{-1}(V(I)) = V(\varphi(I))$.

Pf: P is in the LHS $\iff \varphi^{-1}(P) \in V(I) \iff I \subseteq \varphi^{-1}(P)$ $\iff \varphi(I) \subseteq P \iff P$ is in the RHS. $\square$

We already know that the prime ideals of R/I are in correspondence w/ prime ideals in $\operatorname{Spec} R$ containing I.
In fact:

Claim:
The map $\operatorname{Spec}(R/I) \to \operatorname{Spec}(R)$ is a homeomorphism* of $\operatorname{Spec}(R/I)$ w/ $V(I) \subseteq \operatorname{Spec}(R)$

Check this! (Similar problem on HW)

* a homeomorphism is a continuous, invertible map whose inverse is also continuous.

Ex: Consider the quotient $\mathbb{Z} \to \mathbb{Z}/(2)$. What is the induced map on spec?

$$\operatorname{Spec} \mathbb{Z}/(2) = \{(0)\}$$

$$\operatorname{Spec} \mathbb{Z} = \{(0), (2), (3), (5), \dots\}$$

So we want the pre-image of (0), which is (2).

So $(0) \longmapsto (2)$.

Ex: If $\mathbb{Z} \to \mathbb{Q}$ is the inclusion, what is the map on Spec? $\mathbb{Q}$ is a field so $\operatorname{Spec}\mathbb{Q} = \{(0)\}$, and so $(0) \longmapsto (0)$.

Def: If R is a ring, the quotient by the nilradical

$$R_{red} := R/\mathfrak{N}$$

is called the <u>reduced ring of R</u>.

Since $\mathfrak{N}$ is contained in every prime ideal, we get $V(\mathfrak{N}) = \operatorname{Spec} R$, and from above, the quotient map $R \longrightarrow R_{red}$ induces a homeomorphism

$$\operatorname{Spec} R_{red} \xrightarrow{\cong} \operatorname{Spec} R.$$

(See HW)